

## Chapter1

### § Convergence

For a sequence  $\{x_i\}_{i=1}^{i=\infty}$ , if  $\lim_{i \rightarrow \infty} x_i = x^*$ , then the sequence  $\{x_i\}_{i=1}^{i=\infty}$  is said to be converged to  $x^*$ .

On the other hand, if  $|x_{n+1} - x_n| < \delta$ , then  $\{x_i\}_{i=1}^{i=\infty}$  is also said to be a convergent sequence.

For a sequence  $\{x_i, f(x_i)\}_{i=1}^{i=\infty}$ , if  $\lim_{i \rightarrow \infty} x_i = x^*$  implies  $\lim_{i \rightarrow \infty} f(x_i) = f(x^*)$ ,

then the function  $f(x)$  is continuous at  $x = x^*$ , i.e.,

$$\lim_{\delta x \rightarrow 0} f(x^* - \delta x) = \lim_{\delta x \rightarrow 0} f(x^* + \delta x) = f(x^*) \text{ or}$$

For  $|f(x) - f(x^*)| < \varepsilon$ , there exists a  $\delta$  such that  $|x - x^*| < \delta$ .

Usually, the procedures of solving the equation  $f(x) = 0$  can be expressed as the iteration  $x = g(x)$  which generates the sequence

$\{x_i, g(x_i)\}_{i=1}^{i=\infty}$ . The solution is obtained as  $\lim_{i \rightarrow \infty} x_i = x^*$  or  $\lim_{i \rightarrow \infty} g(x_i) = g(x^*)$ .

Practically, the solution is estimated as  $x_n$  while  $|x_{n+1} - x_n| < \delta$  or

$$|g(x_{n+1}) - g(x_n)| < \delta.$$

### § Differentiation

If  $\lim_{x \rightarrow x^*} \frac{f(x) - f(x^*)}{x - x^*}$  exists, then  $f(x)$  is said to be differentiable at  $x = x^*$ .

Suppose that  $f(x)$  is differentiable at  $x = x^*$ , then  $f(x + \Delta x)$  and

$f(x - \Delta x)$  can be expanded as

$$f(x + \Delta x) = f(x) + f'(x)\Delta x + \frac{1}{2}f''(x)\Delta x^2 + \frac{1}{3!}f'''(x)\Delta x^3 + \dots$$

$$f(x - \Delta x) = f(x) - f'(x)\Delta x + \frac{1}{2} f''(x)\Delta x^2 - \frac{1}{3!} f'''(x)\Delta x^3 + \dots$$

Moreover,

$$f'(x) = \frac{f(x + \Delta x) - f(x)}{\Delta x} - \frac{1}{2} f''(x)\Delta x - \frac{1}{3!} f'''(x)\Delta x^2 - \dots \quad (\text{A})$$

$$f'(x) = \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x} - \frac{1}{3!} f'''(x)\Delta x^2 - \dots \quad (\text{B})$$

Since  $\Delta x$  is small, the expression of (B) is more precise than that of (A).

### § Mean Value

If the function  $f(x) \in C[a, b]$  and  $f(x)$  is differentiable on  $(a, b)$ , then

there exists  $c, d \in (a, b)$  such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{and} \quad \int_a^b f(x) dx = f(d)(b - a)$$

### § Intermediate Value

Suppose that the function  $f(x) \in C[a, b]$  and  $f(a) < f(b)$ , for any constant

$k$  satisfies  $f(a) < k < f(b)$  there exists a  $c \in (a, b)$  such that  $f(c) = k$ .

To find one root of the equation  $f(x) = 0$ , we can try two distinct values

$a$  and  $b$  to satisfy  $f(a)f(b) < 0$ , then the root locates inside  $(a, b)$ .

### § Errors

Consider a number  $y = 0.d_1 \dots d_k d_{k+1} \dots \times 10^n$

There are two ways to express  $y$  at  $k$  decimal digits:

1. Chopping:  $fl(y) = 0.d_1 \dots d_k \times 10^n$
2. Rounding: if  $d_{k+1} < 5$ ,  $fl(y) = 0.d_1 \dots d_k \times 10^n$ ;

$$\text{if } d_{k+1} > 5, \quad fl(y) = 0.d_1 \dots (d_k + 1) \times 10^n$$

There are two types of errors:

$$1. \text{ Absolute error : } \Delta y = y - fl(y)$$

$$2. \text{ Relative error: } \varepsilon = |\Delta y / y|$$

Ex. Evaluate  $f(x) = x^3 - 6.1x^2 + 3.2x + 1.5$  at  $x = 4.71$  using three-digit arithmetic

$$\text{Exact: } x = 4.71, \quad x^2 = 22.1841, \quad x^3 = 104.487111, \quad 6.1x^2 = 135.32301, \\ 3.2x = 15.072;$$

$$f(4.71) = 104.487111 - 135.32301 + 15.072 + 1.5 = -14.263899$$

$$\text{Chopping: } x = 4.71, \quad x^2 = 22.1, \quad x^3 = 104., \quad 6.1x^2 = 134., \quad 3.2x = 15.0;$$

$$f(4.71) = ((104. - 134.) + 15.0) + 1.5 = -13.5$$

$$\text{Rounding: } x = 4.71, \quad x^2 = 22.2, \quad x^3 = 105., \quad 6.1x^2 = 135., \quad 3.2x = 15.1;$$

$$f(4.71) = ((105. - 135.) + 15.1) + 1.5 = -13.4$$

Express  $f(x)$  into the form  $f(x) = ((x - 6.1)x + 3.2)x + 1.5$

Chopping:

$$f(4.71) = ((4.71 - 6.1) \times 4.71 + 3.2) \times 4.71 + 1.5 \\ = ((-1.39) \times 4.71 + 3.2) \times 4.71 + 1.5 = (-6.54 + 3.2) \times 4.71 + 1.5 = -15.7 + 1.5 = -14.2$$

Rounding:  $f(4.71) =$

$$((-1.39) \times 4.71 + 3.2) \times 4.71 + 1.5 = (-6.55 + 3.2) \times 4.71 + 1.5 = -15.8 + 1.5 = -14.3$$

## § Rate of Convergence

Suppose that  $x_{n+1} = g(x_n)$  has a root  $x^*$ . The error of the  $i$ th iteration is

denoted as  $E_i = x_i - x^*$ .

If  $E_n \approx c_1 E_1$ , then the error is called a linear error

If  $E_n \approx c^n E_1$ , then the error is called an exponential error.

Suppose  $\{\alpha_n\}_n \rightarrow \alpha$  and  $\{\beta_n\}_n \rightarrow 0$

If  $|\alpha_n - \alpha| \leq k|\beta_n|$ ,  $n > N$ , then  $\{\alpha_n\}_n \rightarrow \alpha$  with a rate of convergence of

order  $\beta_n$ :  $\alpha_n = \alpha + O(\beta_n)$

Ex. Both  $\alpha_n = \frac{n+1}{n^2}$  and  $\hat{\alpha}_n = \frac{n+3}{n^3}$  converge to 0, however,

$$|\alpha_n| = \left| \frac{n+1}{n^2} \right| < \left| \frac{2n}{n^2} \right| = 2 \left| \frac{1}{n} \right|, \quad |\hat{\alpha}_n| = \left| \frac{n+3}{n^3} \right| < \left| \frac{4n}{n^3} \right| = 4 \left| \frac{1}{n^2} \right|.$$

So,  $\alpha_n = 0 + O(\frac{1}{n})$ ,  $\hat{\alpha}_n = 0 + O(\frac{1}{n^2})$ , which means  $\hat{\alpha}_n$  converges to zero

faster than  $\alpha_n$  does.

## HW 1

1. Let  $f(x) = x^3 - 5.1x^2 + 3.1x + 1.6$ .

a. Evaluate  $f(3.71)$  by calculating  $(3.71)^2$  and  $(3.71)^3$  using four-digit rounding arithmetic.

b. Evaluate  $f(3.71)$  by using the formula  $f(x) = x(x(x - 5.1) + 3.1) + 1.6$ .

c. Compute the absolute and relative errors in parts (a) and (b)

2. Use three-digit chopping arithmetic to compute the sum  $\sum_{i=1}^4 (1/i^2)$  first

by  $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}$  and then by  $\frac{1}{16} + \frac{1}{9} + \frac{1}{4} + \frac{1}{1}$ .

Which method is more accurate, and why?

3. Find the rate of convergence of the function  $\lim_{h \rightarrow 0} \frac{\sin(h) - h \cos(h)}{h} = 0$  as  $h \rightarrow 0$ .